MATHEMATICS BOOKLET

FOR THE NEW FIRST YEAR

BIOCHEMISTRY and BIOMEDICAL SCIENCES

UNDERGRADUATES

UNIVERSITY OF OXFORD

May 2011

-NOTE-

IT IS ABSOLUTELY ESSENTIAL
THAT YOU ARE FAMILIAR WITH

****** ALL ******

THE MATERIAL IN THIS
BOOKLET AT THE START OF THE
MATHEMATICS AND STATISTICS
COURSE FOR
BIOCHEMISTS and BIOMEDICAL SCIENTISTS

University of Oxford
May 2011

Principles of Mathematics Course for First Year Biochemists and Biomedical Scientists.

Why do Biochemists and Biomedical Scientists need Maths?
A knowledge of basic Maths is an essential tool for all sciences, including Biomedicine. All areas of the subject require an ability to manipulate numbers and very simple equations (e.g. making up chemical solutions to the correct concentration, handling enzyme binding data, and carrying out statistical analyses on cell populations).

Why have a Maths Course?
From past experience we have found that amongst new science students there is a far greater spread of ability in Maths than in any other area, and as a result a sizeable minority of students encounter some difficulty with a few areas of the courses. The Maths course is designed to ensure that all first year Biochemists and Biomedical Scientists acquire (or already have) the basic Maths skills required. This means covering only those topics which will be useful for the bulk of the two courses. For most topics the standard required is about equivalent to 'A' Level.

What is the Principles of Mathematics course?
The Principles of Mathematics course consists of 12 lectures and 8 problems classes in the first term, and 4 examples lectures in the second term, of the first year. It is part of a larger course called Principles of Maths and Statistics which runs for the whole of the first year.

What is the purpose of this booklet?
It is not possible during a Biochemistry or Biomedical Sciences course to teach Maths from scratch, so we need to assume that all students arrive with a certain level of skill (mostly in numeracy and very simple algebra). This booklet is designed for two purposes.
(1) If you are confident of your mathematical knowledge you should use it to check that you know all the necessary background maths before you come up.
(2) If you are not confident of your knowledge/ability, use it as a guide to the sort of material you need to cover and to be used in conjunction with a text book. You are **very strongly urged** to work through the booklet, do the problems (answers are included) and to brush up on areas that seem weak. If you encounter difficulties, DON'T PANIC, but ask someone for help (either your school Maths Teacher or your Oxford College Tutor would be good places to start).

Recommended Text Books:

The following books are recommended:
1) 'Foundation Maths. Essential Maths for Students.' by Anthony Croft and Robert Davison. Fourth Edition Published by Prentice Hall (2006) ISBN 0-13-197921-3. Price: £29-99

The following booklet broadly covers the material in Chapters 1 to 14 inclusive and familiarity with it (and Chapters 21, 22, 23 and 24) will be assumed at the start of the Maths Course. The lectures will cover the material in Chapters 16 to 19, 23, and 28 to 32 inclusive, as well as some topics not in the book.

2) Another more advanced and very useful book which we will use in the course is:
'Mathematics for Biological Scientists' by Mike Aitken, Bill Broadbent and Steve Hladky. First Edition Published by Garland Science (2010) ISBN 978-0-8153-4153-4136-9. Price from Amazon: £29-95

Index of the following booklet:

(Chapter numbers in brackets refer to the first recommended text book (fourth edition).)

Thanks to Dr. Robert Esnouf for checking the answers to all the problems.

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Eighth Edition: May 2011

1. BASIC FRACTIONS:

e.g. $\frac{3}{5}$ e.g. $\frac{1}{4}$

3 is the numerator, 5 is the denominator. 1 is the numerator, 4 is the denominator.

(A) When fractions are added and subtracted, we find a common denominator and then add or subtract the numerator.
A common denominator is a number that is a multiple of both denominators. For 3/5 and 1/4 the common denominator is 20, so to add them we express 3/5 as 12/20 (multiplying top and bottom by 4) and 1/4 as 5/20. Then $12/20 + 5/20 = 17/20$.

Exercise 1. Do not use a calculator.
Always express the answers as 'reduced' fractions e.g. 2/4=1/2

(a) $\frac{1}{2} + \frac{1}{3}$ (b) $\frac{1}{4} + \frac{2}{5}$ (c) $\frac{2}{3} + \frac{4}{7}$ (d) $\frac{19}{14} + \frac{1}{6}$

(e) $\frac{13}{16} + \frac{1}{32}$ (f) $\frac{1}{2} - \frac{1}{6}$ (g) $\frac{5}{6} - \frac{1}{12}$ (h) $\frac{3}{10} - \frac{2}{15}$

(i) $\frac{9}{11} - \frac{4}{5}$ (j) $\frac{27}{30} - \frac{1}{5}$ (k) $\frac{2}{13} - \frac{2}{169}$ (l) $\frac{1}{4} - \frac{1}{144}$

(B) When two fractions are multiplied together, their numerators are multiplied together and their denominators are multiplied together.
For example, $\frac{3}{8} \times \frac{5}{6} = \frac{15}{48} = \frac{5}{16}$

Exercise 2. Do not use a calculator.

(a) $\frac{2}{5} \times \frac{3}{8}$ (b) $\frac{11}{15} \times \frac{3}{4}$ (c) $\frac{1}{9} \times \frac{7}{6}$ (d) $\frac{16}{17} \times \frac{1}{4}$

(e) $1\frac{3}{4} \times \frac{1}{3}$ (f) $\left(\frac{3}{4}\right)^2$ (g) $\left(\frac{-3}{7}\right)^2$ (h) $\left(\frac{-1}{2}\right)^3$

(i) $\frac{-5}{3} \times \frac{6}{7}$ (j) $\left(\frac{1}{3}\right)^4$

(C) When we divide by a fraction, we multiply by the reciprocal (inverted or 'turned upsidedown') fraction.
For example, $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3} = \frac{14}{15}$

Exercise 3. Do not use a calculator.

(a) $\frac{1}{2} \div \frac{1}{3}$ (b) $\frac{4}{9} \div \frac{2}{9}$ (c) $\frac{4}{9} \div \frac{1}{2}$ (d) $\frac{-4}{9} \div 2$

(e) $6\frac{1}{3} \div 3\frac{1}{4}$ (f) $\left(\frac{18}{19}\right) / \left(\frac{1}{3}\right)$ (g) $\left(\frac{5}{6}\right) / \left(\frac{15}{9}\right)$

(h) $\left(\frac{1}{6}\right) / \left(\frac{1}{5}\right)$ (i) $\left(\frac{-1}{3}\right) / \left(\frac{-1}{15}\right)$ (j) $\frac{99}{100} \div 33$

Answers:
Exercise 1: (a) 5/6, (b) 13/20, (c) $1\ \frac{5}{21}$, (d) $1\frac{11}{21}$, (e) 27/32, (f) 1/3, (g) 3/4, (h) 1/6, (i) 1/55, (j) 7/10, (k) 24/169, (l) 35/144.

Exercise 2: (a) 3/20, (b) 11/20, (c) 7/54, (d) 4/17, (e) 7/12, (f) 9/16, (g) 9/49, (h) −1/8, (i) $-1\frac{3}{7}$, (j) 1/81.

Exercise 3: (a) $1\frac{1}{2}$, (b) 2, (c) 8/9, (d) −2/9, (e) $1\frac{37}{39}$, (f) $2\frac{16}{19}$, (g) 1/2, (h) 5/6, (i) 5, (j) 3/100.

2. BASIC ALGEBRA

(A) Algebraic representation. Representing constants by symbols, a, b, c....... up to m.
Representing variables by symbols, n,........x, y, z.
This is simply convention to avoid confusion. (o is never used as a symbol since it can be confused with 0).
e.g.(i)

$$5 = 2 + 3$$

can be represented as

$$z = x + y$$

where z=5, x=2 and y=3.
e.g.(ii)

$$3z = 4x + 6y$$

Here possible values would be z=5, x=3/4 and y=2 to satisfy the equation. The ‘coefficient’ of z is said to be 3.
e.g.(iii)

$$az = bx + cy$$

Here there are a number of possible solutions, since a, b and c can take any values. The ‘coefficient’ of z is said to be a.

(B) Introducing Indices.

$$4 \times 4 \times 4 = 4^3 \quad \text{or in symbols } a \times a \times a = a^3$$

This is called the ‘exponential’ form. 3 is the ‘power’ and a is the ‘base’.

Exercise 1:
Write in exponential form, identifying the base and exponent: (a) 25 (b) 81 (c) 169 (d) 27. What are (e) 4^4 (f) 4^3 (g) 4^2 (h) 4^1 (i) 4^0 (j) 2^3 (k) 3^2?

(C) Brackets. Everything inside the brackets is multiplied by what is immediately outside the bracket:

$$5(x + 4y - 7) = 5x + 20y - 35$$

(D) Equations. Statement that 2 quantities are equal. (To be distinguished from an ‘expression’ which does not contain an equals sign).

$$\text{e.g. } 5x - 1 = 6y + 3$$

Exercise 2:
For the equation: $5x + 3y - 15bc + 17xyz + 7 + 19x^2 = Q$ write down (a) the terms in x, (b) the constant coefficients, (c) the coefficient of x^2 and (d) the coefficient of z.

(E) Formulae. An equation connecting 2 or more variable quantities. e.g. To calculate the average cost per person of feeding a household for a week, we divide the total food bill by the number of people living in the household:
Let V = average cost, T = total bill, and n = number of people, so $V = \frac{T}{n}$

Exercise 3:
Write down formulae to represent the following relationships. Remember to define your variables (let.....).
(a) The radius of a circle is half the diameter.
(b) The volume of a sphere is $4\pi/3$ times the radius cubed.
(c) A flea can jump 100 times higher in *cm* than its weight in *grams*.
(d) The distance cycled by my daughter every week is: cycling to school and back each weekday, to her friend's house twice a week and to the swimming pool once a week.
(e) The annual consumption of blades of grass of a horse if it eats for a quarter of the day and consumes as many blades of grass in an hour as hairs on its back.
(f) My net pay at the end of the month is the gross amount with my pension and national insurance contributions subtracted, and then with 24% tax deducted from what remains after taking into account my tax free allowance.

(F) Substitution. If we know x and y we can put them into our formulae and calculate numerical values.
To calculate a numerical value from a formula with n unknown quantities, $(n-1)$ of them must be specified. Beware of mixing units!

Exercise 4:
If $x = 6$, $y = 3$ and $z = -5$ find:

(a) x^2y (b) $z^2 + x$ (c) $3xy + 5zy - 6xz$ (d) $z\,(y + 2x)$

(e) $\dfrac{xyz}{x + y + z}$

(f) Repeat for $x = \frac{1}{3}$, $y = \frac{1}{4}$, $z = \frac{1}{5}$.
(g) Find P where $P = L + 2W$ when $L = 1m$ and $W = 10cm$
(h) Find V where $V = \pi l^2 d$ when $l = 10cm$ and $d = 15cm$

(i) Find E where $E = \frac{hc}{\lambda}$ with $h = 6.62 \times 10^{-34} Js$, $c = 3.0 \times 10^8 ms^{-1}$ and $\lambda = 400nm$.
(j) Find v when $x = 2$ and $y = 3$ where:

$$v^2 = 3\left(\frac{1}{x} - \frac{1}{y}\right)$$

(G) Like and unlike terms. Can simplify like terms by combining them together but can not combine unlike terms. e.g.

$$13x + xy - 3y - 5x + 20y - 5xy = 8x + 17y - 4xy$$

since we can collect and combine terms in x, y amd xy separately to simplify the expression.

Exercise 5:
Simplify by collecting like terms:

(a) $6x + 15x + 7y + 9y$ (b) $15x - 6x - 3y + 10y$ (c) $-15x - 6x - 15y - 6y$

(d) $-15y+4x-6y+7y+3x$ (e) $6x^2+15x^2+7x^2$ (f) $3xy+4z-yx+y^2-z+3zy-3y^2$

Simplify by cancelling terms:

(g) $\frac{6z}{z^2}$ (h) $\frac{5xy}{xy}$ (i) $\frac{-10ab}{-5bc}$ (j) $\frac{y}{xy}$ (k) $\frac{15xy^2z}{5x^2y^2z^2}$

(H) Manipulating Indices.

[Note: both '.' and '×' are used to represent the multiplication symbol.]

(i) When multiplying the powers of a number together, we add the indices:

$$b^2.b^4 = (b.b).(b.b.b.b) = b^6$$

(ii) When dividing powers of a number, we subtract the indices.
(iii) When taking powers of powers, we multiply the indices together.
(iv) If a product is raised to a certain power, *each part* of the product is raised to that power.
(v) If a quotient is raised to a certain power, both the numerator and denominator are raised to that power.

$$\text{e.q.} \quad b^6 = \frac{b^8}{b^2} = \left(b^2\right)^3 = \left(a^6c^6\right) = (ac)^6 \quad \text{where} \quad b = ac$$

.

$$b^6 = \left(\frac{m^2}{n^2}\right)^3 = \left(\frac{m^6}{n^6}\right) = \left(\frac{m}{n}\right)^6 \quad \text{where} \quad b = \frac{m}{n}$$

N.B. $a^0 = 1$ and $a^1 = a$

Exercise 6:
Simplify:

(a) $5^4.5^2$ (b) $a^{12}.a^{11}$ (c) $\left(3^3\right)^2$ (d) $\left(4x^5\right)^3$

(e) $\dfrac{x^3.x^2}{x^4}$ (f) $\dfrac{a^{13}}{a^{12}}$ (g) $\dfrac{12^2 \times 3}{2^3 \times 3^3}$ (h) $\dfrac{30^2 \times 5^3}{2^2 \times 3 \times 5^5}$

(i) $(3y)^4$ (j) $\left(s^3r^4\right)^3$ (k) $(ab)^3 \,.\, \left(a^2b^2\right)^2$ (l) $2a + b^2 + a + a^2$

(m) $\left(\dfrac{2x^2y}{ab^2c}\right)^2 . \left(\dfrac{4a^2bc}{3xy}\right)^3$ (n) $\dfrac{a^3b^4c^2.abc}{a^2b^5c^3}$ (o) $\left(\dfrac{b^4}{a^2}\right)^3$ (p) $\dfrac{a^4b^2}{a^2b^3}$

(J) Combining square and higher order roots.
The same rules apply, we just have to deal with fractions as well as integers.

Square root: $\sqrt{a} = a^{1/2}$ Cube root: $\sqrt[3]{a} = a^{1/3}$

Fourth root: $\sqrt[4]{a} = a^{1/4}$

General algebra of roots:

(i) $\sqrt{a} \times \sqrt{a} = a$ (ii) $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = (ab)^{1/2}$

(iii) $\dfrac{a}{\sqrt{a}} = \sqrt{a} = a^{1/2}$ (iv) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

(v) To rationalise a denominator containing a single square root, multiply numerator and denominator by that square root. e.g.

$$\frac{b}{\sqrt{a}} = \frac{b\sqrt{a}}{\sqrt{a}\sqrt{a}} = \frac{b\sqrt{a}}{a}$$

Exercise 7.
Simplify:

(a) $\dfrac{x^{1/3}.x^{-4/3}}{x^{-1}}$ (b) $\dfrac{x^{1/2}}{x^{-1/2}}$ (c) $\dfrac{\left(x^{1/2}\right)^4}{x^3 \times x^2}$ (d) $\dfrac{a^4}{a^{-2}}$

(e) $\dfrac{(\sqrt{x})^6}{x^5}$ (f) $\dfrac{y^{1/3}}{y^{-2/3}}$ (g) $\dfrac{x^2.\sqrt{x}}{x^{3.5}}$

IMPORTANT NOTE:

(i) $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$

(ii) $\dfrac{1}{\sqrt{a}+\sqrt{b}} \neq \dfrac{1}{\sqrt{a+b}}$

Answers:
Exercise 1: (a) 5^2 (b) 9^2 or 3^4 (c) 13^2 (d) 3^3 (e) 256 (f) 64 (g) 16 (h) 4 (i) 1 (j) 8 (k) 9

Exercise 2: (a) 5 $17yz$ (b) 7, $-15bc$ (c) 19 (d) $17xy$

Exercise 3: (a) $r = d/2$ (b) $V = 4\pi r^3/3$ (c) $H = 100x$ (d) $D = 5x + 2y + z$ (e) $C = 2190h$ (f) $T = 0.76(G - P - I - A) + A$ (T=net pay, G=gross, P=pension, I= nat insurance, A=tax free allowance.)

Exercise 4: (a) 108 (b) 31 (c) 159 (d) -75 (e) -22.5 (f) 1/36, 28/75, 1/10, 11/60, 1/47,

(g) $120cm$ (h) 4710 cm^3 (i) $4.965 \times 10^{-19}J$ (j) $1/\sqrt{2}$

Exercise 5: (a) $21x + 16y$ (b) $9x + 7y$ (c) $-21x - 21y$ (d) $-14y + 7x$ (e) $28x^2$ (f) $2xy + 3z - 2y^2 + 3zy$ (g) $6/z$ (h) 5 (i) $2a/c$ (j) $1/x$ (k) $3/xz$

Exercise 6: (a) 5^6 (b) a^{23} (c) 3^6 (d) $64x^{15}$ (e) x (f) a (g) 2 (h) 3 (i) $81y^4$ (j) s^9r^{12} (k) $(a.b)^7$ (l) $3a + b^2 + a^2$ (m) $256a^4cx/27by$ (n) a^2 (o) b^{12}/a^6 (p) a^2/b

Exercise 7: (a) 1 (b) x (c) $1/x^3$ (d) a^6 (e) $1/x^2$ (f) y (g) $1/x$

3. FACTORISATION.

Factorisation is the breaking down of algebraic terms into components that have been multiplied together.

(A) Product of $(a+b)(c+d)$

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$$

i.e. 4 multiplications and 4 resulting terms:
Effectively:

$$(a+b)(c+d)$$

(B) Quadratic expressions. (Those containing a term in x^2.)
e.g. $(x+1)(x+2) = x^2 + 3x + 2$
Generally:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad (\text{N.B.} \neq \mathrm{a}^2 + \mathrm{b}^2)$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a-b)(a+b) = a^2 - b^2$$

Exercise 1.
Simplify:

(a) $-3(z-y)$ (b) $4(2y-6z)$ (c) $-y(4y-8/y)$ (d) $2^2(x/12 - y^2/24)$

Multiply out:

(e) $(x-4)\,(y+3)$ (f) $(x+2)^2$ (g) $(x-4)^2$ (h) $(3x+1)\,(3x-1)$

(i) $6\,(4x+2y)\,(c+d)$

(C) Binomial Series.
A binomial series is the series of terms resulting from an expansion of expressions which have the form:

$$(1+x)^n = 1 + \frac{nx}{1} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ...$$

N.B. 4.3.2.1 = 4! (known as '4 factorial') and $1.2.3......n(n-1).n = n!$
If n is a positive integer, this series has $(n+1)$ terms, the series will terminate (i.e. it ends) and the expression is true for *any* value of x.

The binomial expansion can be generalised for the case where n is a positive integer:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ...b^n$$

Exercise 2.
Use the binomial theorem to find the first 4 terms of:

(a) $(1+x)^7$ (b) $(1+3x)^4$

(c) $(4+x)^6 = 4^6\left(1+\frac{x}{4}\right)^6$

(D) Factors.
Often we want to reverse the multiplication process of (A) and find the 'factors' which give the product. We find the 'highest common factor' in the terms:
e.g.(i) $-5xyz - 10x^2yz^3 + 20xy^3z^2$ can be 'factorised' to be:
$5xyz(4y^2z - 1 - 2xz^2)$ where $5xyz$ is the factor common to every term.
e.g.(ii) $ac + ad + bc + bd$ has no factor which is common in all 4 terms, but we can write:
$a(c+d) + b(c+d) = (a+b)(c+d)$

(E) Factorising Quadratic Expressions.
(1) Those where the coefficient of x^2 is 1:
$x^2 + bx + c = (x+g)(x+h)$ where $c = g \times h$ and $b = g + h$
i.e. g and h are the factors of the constant term (which is c) whose algebraic sum is equal to the coefficient of x (which is b).
e.g. $(x+4)(x+6) = x^2 + 10x + 24$
The factors of 24 are (2 and 12) or (6 and 4) or (1 and 24). The coefficient of x is 10, which is the sum of 6 and 4. Hence the factorisation to $(x+4)(x+6)$. The answer can be checked by multiplying the brackets out again.

Exercise 3.
Factorise, if possible:

(a) $rx + sx + tx + ux$ (b) $2\pi r + \pi r^2$ (c) $12x + 15y$

(d) $cx + dy + dx + cy$ (e) $x^2 + dx + cx + cd$ (f) $x^2 + 12x + 27$

(g) $x^2 + 14x + 24$ (h) $x^2 - 4x - 12$ (i) $x^2 - 4x + 3$

(j) $2x + 15 - x^2$ (k) $6x - 8 - x^2$ (l) $2x^4y^3 + 6x^2y^5 - 8x^3y^4$

(2) Those quadratic expressions where the coefficient of x^2 is NOT 1.
i.e. of the form:

$$ax^2 + bx + c$$

e.g. $(3x + 2)(2x + 5) = 6x^2 + 4x + 15x + 10 = 6x^2 + 19x + 10$
We want to factorise this to give an expression of the form:

$$(qx + r)(px + s) = (q \times p)x^2 + (qs + rp)x + rs = ax^2 + bx + c$$

So: $c = rs$, $b = qs + pr$, $a = qp$.
Thus: $ca = qprs$ and among the factors of ca are (qr and ps), (qp and rs) and (pr and qs). It is the last of these pairs which we want. The method is thus to *find the pair of factors of ac which will add up to b.*
For the above example, $ac = 60$ which has factors (10 and 6), (15 and 4), (2 and 30), (12 and 5), (3 and 20) and (1 and 60).
Since $b = 19$ the pair required is (15 and 4), so that $4 = rp$ and $15 = qs$. The factors of 4 and 15 will now give r, p, q and s individually.
So we can write:

$$6x^2 + 4x + 15x + 10 = 2x(3x + 2) + 5(3x + 2) = (3x + 2)(2x + 5)$$

Exercise 4.
Factorise:

(a) $3x^2 + 5x + 2$ (b) $10x^2 + 24x + 8$ (c) $6x^2 + 7x + 2$

(d) $6x^2 - 7x + 2$ (e) $6x^2 + x - 2$ (f) $6x^2 - x - 2$

(E) Difference of 2 squares.
$x^2 - 9$ can be factorised into a sum and a difference:

$$x^2 - 9 = (x + 3)(x - 3)$$

In general: $x^2 - a^2 = (x + a)(x - a)$

Exercise 5.
Factorise:

(a) $x^2 - 81$ (b) $x^2 - 36$ (c) $x^2 - 121$ (d) $5x^2 - 125$

N.B. The SUM of 2 squares has NO *real* factors: e.g. $x^2 + 9$. Imaginary numbers are required to factorise this, and these will not be covered here.

(F) Perfect Squares.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

If the quadratic does not have exactly the above form, we can sometimes add another term to 'complete the square'.
e.g. $x^2 + 6x + 3$ can be made a perfect square by adding 6.
The constant coefficient which 'completes the square' is the square of half the coefficient of x.
So: $\quad x^2 + 6x + 3 = x^2 + 6x + 9 - 6 = (x + 3)^2 - 6$

Exercise 6.
Factorise the following perfect squares:

(a) $\quad x^2 + 12x + 36$ $\qquad$ (b) $\quad x^2 - 8x + 16$ $\qquad$ (c) $\quad 9x^2 + 12x + 4$

Write down the term which would complete the square in the following expressions and then factorise them:

(d) $\quad x^2 + 4x + 2$ $\qquad$ (e) $\quad 4x^2 + 12x$ $\qquad$ (f) $\quad x^2 + 10x + 15$

Answers:
Exercise 1: (a) $3(y - z)$ (b) $8(y - 3z)$ (c) $4(2 - y^2)$ (d) $(x - y^2/2)/3$ (e) $xy - 4y + 3x - 12$ (f) $x^2 + 4x + 4$ (g) $x^2 - 8x + 16$ (h) $9x^2 - 1$ (i) $24cx + 12cy + 24dx + 12dy$

Exercise 2: (a) $1+7x+21x^2+35x^3$ (b) $1+12x+54x^2+108x^3$ (c) $4^6(1+3x/2+15x^2/16+5x^3/16$.

Exercise 3: (a) $(r+s+t+u)x$ (b) $\pi r(r+2)$ (c) $3(4x+5y)$ (d) $(c+d)(x+y)$ (e) $(x+c)(x+d)$ (f) $(x+3)(x+9)$ (g) $(x+2)(x+12)$ (h) $(x-6)(x+2)$ (i) $(x-1)(x-3)$ (j) $(5-x)(3+x)$ (k) $(2-x)(x-4)$ (l) $2x^2y^3(y-x)(3y-x)$.

Exercise 4: (a) $(3x+2)(x+1)$ (b) $(2x+4)(5x+2)$ (c) $(2x+1)(3x+2)$ (d) $(2x-1)(3x-2)$ (e) $(2x-1)(3x+2)$ (f) $(2x+1)(3x-2)$

Exercise 5: (a) $(x + 9)(x - 9)$ (b) $(x + 6)(x - 6)$ (c) $(x + 11)(x - 11)$ (d) $5(x + 5)(x - 5)$

Exercise 6: (a) $(x+6)^2$ (b) $(x-4)^2$ (c) $(3x+2)^2$ (d) $2,\ (x+2)^2$ (e) $9,\ (2x+3)^2$ (f) $10,\ (x+5)^2$

4. EQUATIONS.

(A) Manipulation of equations:
(i) An equation is unchanged by adding or subtracting the same number on both sides, e.g. $3x + 1 = 6$ is equivalent to $3x + 6 = 11$ (adding 5 to both sides),
(ii) An equation is unchanged by multiplying or dividing both sides by the same number, e.g. $5x = 15$ is equivalent to $20x = 60$ (multiplying both sides by 4).
(iii) If $A = B$ and $B = C$ then $A = C$, e.g. $E = h\lambda/c$ and $h\lambda/c = h\nu$, then $E = h\nu$
(iv) If $A = B$ and $A = C$ then $B = C$ e.g. $V = RT/P$ and $V = M/\rho$, then $RT/P = M/\rho$

(B) Solving linear equations
In the above example, the value of x could be deduced from the equations. This is known as 'solving', or 'finding the solution' of the equation.

Exercise 1. By applying the above rules of manipulation, solve these equations for x.

(a) $x + 6 = 15$ (b) $3x + 5 = 17$ (c) $16x - 4 = 28$ (d) $9x - 2 = -6$

(e) $\frac{15}{x} = 3$ (f) $\frac{x}{9} = 10$ (g) $\frac{x}{6} = \frac{5}{3}$ (h) $\frac{42}{x} = \frac{6}{7}$

(i) $15 - x = -3$ (j) $4(x + 7) = 6(x - 3)$ (k) $6(x + 1) = 18$

(l) $\frac{5x}{4} + \frac{1}{2} = 3$ (m) $\frac{3x - 1}{4} = \frac{3}{4}$ (n) $\frac{x}{4} + 6 = \frac{x}{3} - 4$

(o) $\frac{x - 6}{7} = \frac{x + 4}{10}$ (p) $\frac{6x + 1}{4} - \frac{2x - 1}{3} = \frac{8 - 3x}{2}$

(C) An aside: The modulus (or absolute value) of a number.
This is the size of the number, x, ignoring its sign and is written $|x|$. e.g.$|4| = 4$ and $|-4| = 4$.
For 'mods',

(i)
$$|ab| = |a| \times |b|$$

(ii)
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad (if\ b \neq 0)$$

(iii)
$$|a^n| = |a|^n$$

(D) Manipulation of formulae.
The formula $y = 3x + 2$ is an 'explicit' formula for y, as it explains how to calculate y without rearrangement. y is said to be the 'subject' of the equation.
However, the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ is said to be 'implicit', since the relationship between f, u and v is implied rather than explicit. e.g. here $f = \frac{uv}{u+v}$ gives the explicit relationship.

Exercise 2.
Manipulate the following equations to change the subject as specified:

(a) $y = 3x + 2$ for x (b) $x = u + at$ for t (c) $y = cx - a$ for x

(d) $F = mg + m\omega^2 r$ for r (e) $E = hc/\lambda$ for λ (f) $y = bx^3 + d$ for x

(g) $\dfrac{x^2}{\sqrt{a}} = 3q$ for a (h) $y = \dfrac{3x-2}{3x+1}$ for x (i) $a = \dfrac{2bc^2}{b+c}$ for b

(j) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{c-2}{x}$ for x (k) $y = \dfrac{1-x^2}{1+x^2}$ for x (l) $\dfrac{a}{A} = \sqrt{\dfrac{b+d}{b-d}}$ for b

(E) Elimination. If we know how two variables (say x and y) are related to a third variable (say z) we can eliminate z and find out how x and y are related. This is called eliminating a 'parameter'. e.g. The equations:

$$P = \frac{RT}{V} \quad \text{and} \quad \rho = \frac{M}{V}$$

can give us a relationship between P and M, since $V = M/\rho$ and $V = RT/P$ so V can be eliminated and:

$$\frac{M}{\rho} = \frac{RT}{P} \qquad \text{and} \quad \text{then} \quad P = \frac{RT\rho}{M}$$

The method here consists of the following steps:
(i) Make the term to be eliminated the subject of both equations.
(ii) Use Section 4(A) (iii) to set the equations equal.
(iii) Manipulate the equation to give the required subject.

Exercise 3
(a) Eliminate b, making y the subject, from $x = 3y + 5b$ and $b = 6xy$
(b) Eliminate ν, making λ the subject, from $E = h\nu$ and $\lambda\nu = c$
(c) Eliminate t, making u the subject, from $v = u + at$ and $s = ut + \frac{1}{2}at^2$

(d) Eliminate r, making A the subject, from $V = 4\pi r^3/3$ and $A = 4\pi r^2$

Answers:
Exercise 1: (a) $x = 9$ (b) $x = 4$ (c) $x = 2$ (d) $x = -4/9$ (e) $x = 5$ (f) $x = 90$ (g) $x = 10$ (h) $x = 49$ (i) $x = 18$ (j) $x = 23$ (k) $x = 2$ (l) $x = 2$ (m) $x = 4/3$ (n) $x = 120$ (o) $x = 29\frac{1}{3}$

(p) $x = 1\frac{13}{28}$

Exercise 2: (a) $x = (y-2)/3$ (b) $t = (x-u)/a$ (c) $x = (y+a)/c$ (d) $r = (F - mg)/mw^2$

(e) $\lambda = hc/E$ (f) $x = \sqrt[3]{(y-d)/b}$ (g) $a = x^4/9q^2$ (h) $x = (y+2)/3(1-y)$ (i) $b = ac/(2c^2 - a)$ (j) $x = ab(c-2)/(a+b)$ (k) $x = \sqrt{(1-y)/(1+y)}$ (l) $b = (A^2+a^2)d/(a^2-A^2)$

Exercise 3: (a) $y = x/(30x+3)$ (b) $\lambda = hc/E$ (c) $u = \sqrt{v^2 - 2as}$ (d) $A = \sqrt[3]{36\pi V^2}$

5. QUADRATIC EQUATIONS.

We can solve the equations $xy = 0$ or $x + a = 0$.

(A) Zero Products

If the product of two numbers is zero, at least one of them must be zero: $3 \times 0 = 0$, $a \times 0 = 0$, $0 \times 0 = 0$ and $ab = 0$ implies that either $a = 0$ or $b = 0$ or both a and b are 0. Similarly if $abcd = 0$ at least one of a, b, c and d must be zero.

(B) Solution of Quadratic Equations.

Such equations are usually of the form:

$$ax^2 + bx + c = 0$$

We must solve for the values of x which satisfy the equation, called finding the 'roots' of the equation. For an equation containing x^2 there are **ALWAYS** 2 roots (but they may sometimes be the same). The roots may be positive, negative or complex numbers. Complex numbers are ones which have an imaginary part.

Case (1). If the constant term (c) is missing,
$ax^2 + bx = 0 = x(ax + b) = 0$ which implies that either $x = 0$ or $(ax + b) = 0$ (i.e. $x = -b/a$).
e.g. (i) $x^2 + 4x = 0 = x(x + 4)$ so $x = 0$ or $(x + 4) = 0$ (i.e. $x = -4$).

Case (2). If the term in x (b) is missing,

$ax^2 + c = 0$, so $ax^2 = -c$ and thus $x = \pm\sqrt{-c/a}$
If c/a is a negative number, the square root of $-c/a$ is a real number.

e.g. (ii) $x^2 - 16 = 0$ so $x^2 = 16$ and thus $x = \pm 4$. Here $c/a = -16/1$, so $\sqrt{(-)(-)16} = 4$
If c/a is a positive number, the square root of $-c/a$ is not a real number and the equation is said to have 'imaginary' roots.

Case (3). All terms are present.
There are two alternative methods of finding the roots of the quadratic: factorisation (see Section 3 of these notes) and by the quadratic equation (see section (C) below).
By factorisation:
e.g. (iii) $x^2 + 5x + 6 = 0 = (x + 2)(x + 3)$ so the roots are $x = -2$ or -3.

Exercise 1. Find the roots of the following equations:

(a) $x^2 - 6x = 0$	(b) $4x^2 - 2x = 0$	(c) $x^2 - x = 0$
(d) $25x^2 - x = 0$	(e) $x^2 - 49 = 0$	(f) $16 - 4x^2 = 0$
(g) $100 + x^2 = 0$	(h) $13x^2 - 21 = 0$	(i) $x^2 + 9x + 20 = 0$
(j) $x^2 + 4x - 21 = 0$	(k) $4x^2 + 6x + 2 = 0$	(l) $x^2 - 5x - 50 = 0$

(C) The Quadratic Formula. An alternative method is to manipulate the generalised quadratic equation to make x the subject (see Section 4 of this booklet for equation manipulation). It is not necessary to know the derivation but you should be able to follow it based on the previous sections of this booklet.

$$ax^2 + bx + c = 0$$

dividing through by a : $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

and rearranging, $$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Completing the square by adding $(b/2a)^2$ to both sides (see Section 3(F)):

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Factorising the perfect square gives : $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Taking the square root of both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

So $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: If $b^2 - 4ac > 0$ the quadratic has **two, distinct, real roots**.
Case 2: If $b^2 - 4ac = 0$, the square root term disappears and the quadratic has **two equal roots**.
Case 3: If $b^2 - 4ac < 0$ the square root term is imaginary and the quadratic has **no real roots**. The roots are complex numbers.

Exercise 2.
By using the quadratic formula, find the roots of:

(a) $5x^2 - 13x + 6 = 0$ (b) $10x^2 + x - 2 = 0$ (c) $3 - x^2 - 2x = 0$
(d) $x^2 + x + 12 = 0$ (e) $4 - 2x - 6x^2 = 0$ (f) $2x^2 + 4x + 2 = 0$

Answers:
Exercise 1: (a) $x = 0$ or 6 (b) $x = 0$ or 1/2 (c) $x = 0$ or 1 (d) $x = 0$ or 1/25 (e) $x = \pm 7$
(f) $x = \pm 2$ (g) $x = \pm\sqrt{-100}$ i.e. roots are imaginary. (h) $x = \pm\sqrt{21/13}$ (i) $x = -5,\ -4$
(j) $x = -7,\ 3$ (k) $x = -1/2,\ -1$ (l) $x = -5,\ 10$

Exercise 2: (a) $x = 2,\ 3/5$ (b) $x = -1/2,\ 2/5$ (c) $x = -3,\ 1$ (d) imaginary roots (e) $x = -1,\ 2/3$ (f) $x = -1,\ -1$

6. SIMULTANEOUS EQUATIONS.

Sometimes we have 2 equations which relate linear combinations of x and y, which we can solve for unique values of x and y (which is a pair of numbers x and y). These are called 'simultaneous' equations. e.g.

$$\text{(i)} \qquad 10x + y = 105$$
$$\text{(ii)} \qquad x - y = 5$$

The problem is equivalent to finding the point where 2 straight lines intersect, since a straight line can be parameterised by a linear equation in x and y.

(A) Method of elimination. The above pair of equations have the same coefficient of y but of opposite sign. If we add the equations together, y will be eliminated, and we get:

$$11x = 110 \qquad \text{so } x = 10$$

Substituting $x = 10$ into equation (ii) above, gives $10 - y = 5$ and $y = 5$.
If necessary one equation can be multiplied through by a constant factor to give the same coefficients of x or y as the other equation. The equations can then be added or subtracted from each other as appropriate. e.g.

$$\text{(i)} \qquad 5a - 4b = 6$$
$$\text{(ii)} \qquad a + b = 3$$

Multiply every term in equation (ii) by 4: $4a + 4b = 12$
and add it to equation (i): $9a = 18$ so $a = 2$.
Thus the solution is $a = 2$ and $b = 1$. We can check this by substituting these values back into equations (i) and (ii):
(i) $5 \times 2 - 4 \times 1 = 6$ as expected and (ii) $2 + 1 = 3$ as expected.
It is usually necessary to multiply each of the pair of equations by a **different** factor for the elimination method.

Exercise 1.
Solve the following pairs of equations for x and y.

(a) $20x + y = 81 \qquad 2x - y = 7$
(b) $2x + 5y = 16 \qquad 5x + 2y = 19$
(c) $2x - 7y = 8 \qquad 4x - 6y = 0$
(d) $2y - x = 1 \qquad 5y + \frac{x}{2} = 1$
(e) $2y - 6x = 6 \qquad 20y + 12x = 0$
(f) $3y + 2x = 9 \qquad 5x - y = -3$

(B) Solution by substitution. An alternative method to elimination is to manipulate the equations to make x or y the subject of both and equate them.
e.g. $5x + 5y = 10$ (i) and $2x - 3y = 14$ (ii)
Rearranging (i): $5x = 10 - 5y$ so $x = (10 - 5y)/5 = 2 - y$ (iii)
Rearranging (ii): $2x = 14 + 3y$ so $x = (14 + 3y)/2$ (iv)
Equating (iii) and (iv):

$$2 - y = \frac{14 + 3y}{2}$$

$$4 - 2y = 14 + 3y \quad \text{so} \quad 4 - 14 = 2y + 3y \quad \text{thus} \quad -10 = 5y \quad \text{and} \quad y = -2$$

This value for y is now substituted back into (iii), giving $x = 4$.

Exercise 2.
Solve the following pairs of equations for x and y.

(a) $4y + x = 15$ $9y - x = 37$ (b) $3x + 2y = 7$ $2x + 3y = 3$
(c) $2y - 2x = 6$ $3y + 3x = -3$ (d) $2x - 20y = 15$ $20x + 4y = 48$
(e) $3y - 6x = 36$ $13y + 3x = -18$ (f) $3y - 10x = 2$ $15y + 50x = 0$
(g) A mixed pile of 2kg and 4kg bricks weighs a total of 1000kg and contains 300 bricks. How many of each weight brick are there?
(h) A car averages 28.3 miles/gallon in town and 32.4 miles/gallon on the motorway. On a recent 300 mile trip, fuel consumption averaged 31.2 miles/gallon. How many miles of the journey were on the motorway?

(C) Solution if one of the equations is not linear.
If the equations contain terms in x^2, y^2 or xy, the method of substitution must be used. The resulting equation can then be factorised (see Section 3) and will usually give more than one pair of solutions. The problem is equivalent to finding the points of intersection of a line (represented by the linear equation) and a curve (represented by the non-linear equation). e.g.
$x^2 + y^2 = 13$ (1) and $x - y = 5$ (2).
Substitute $x = 5 + y$ from (2) into (1): $(5 + y)^2 + y^2 = 13 = 25 + y^2 + 10y + y^2$
Now we must factorise:
$2y^2 + 10y + 12 = 0 = y^2 + 5y + 6 = (y + 2)(y + 3)$ so $y = -2$ or -3.
Putting $y = -2$ into (2) gives $x = 3$ and putting in $y = -3$ gives $x = 2$.

Exercise 3.
Solve the following pairs of equations for both pairs of x and y.
(a) $x^2 + y^2 = 2$ $2x - y = 1$ (b) $y^2 - x^2 = 7$ $4x - 3y = 0$
(c) $x^2 + 2x + y^2 = 17$ $5x - 3y = 1$ (d) $y^2 + xy = 6$ $5y + 2x = 15$
(e) $x^2+y^2+3x-2y = 4$ $13x-6y = 1$
(f) $25x^2 + 9y^2 - 5x - 3y = 6$ $5x - 3y = 2$

Sets of equations which have no solutions are :
Case 1: The equations of two parallel straight lines which have different intercepts with the axes. e.g.

(i) $x + y = 3$ and (ii) $x + y = 4$

Case 2: The equations of a line and a curve where the curve does not intercept the line. e.g.

(i) $y = 5x - 3$ and (ii) $x^2 + y^2 = 4$

((ii) is the equation of a circle of radius 2 and centred at the origin.)

Answers:
Exercise 1: (a) $x = 4,\ y = 1$ (b) $x = 3, y = 2$ (c) $x = -3, y = -2$ (d) $x = -1/2, y = 1/4$ (e) $x = -5/6, y = 1/2$ (f) $x = 0, y = 3$

Exercise 2: (a) $x = -1, y = 4$ (b) $x = 3,\ y = -1$ (c) $x = -2, y = 1$ (d) $x = 5/2, y = -1/2$ (e) $x = -6, y = 0$ (f) $x = -1/10, y = 1/3$ (g) $x + y = 300$ and $2x + 4y = 1000$, 100 2kg bricks and 200 4kg bricks. (h) $x + y = 300$ and $\frac{x}{28.3} + \frac{y}{32.4} = \frac{300}{31.2}$, 212.2 miles on the motorway.

Exercise 3: (a) $x = 1, y = 1$ or $x = -1/5, y = -7/5$ (b) $x = 3, y = 4$ or $x = -3, y = -4$ (c) $x = 2, y = 3$ or $x = -38/17 = -2\frac{4}{17}, y = -69/17 = -4\frac{1}{17}$, (d) $x = 5, y = 1$ or

$x = -5/2, y = 4$ (e) $x = 1, y = 2$ or $x = -131/205 = -0.64, y = -1\frac{113}{205} = -1.55$ (f) $x = 0, y = -2/3$ or $x = 3/5, y = 1/3$

7. SOME TRIGONOMETRY PROBLEMS.

This section does not summarise the information you need to do the problems: it consists of problems alone. If you have any difficulties with them, you are urged to consult the textbook recommended at the beginning of this booklet as an alternative to your GCSE notes.

Exercise 1. By convention 2π radians $=$ $360°$. Write down the angle in radians (as fractions of π) of:
(a) $135°$ (b) $67.5°$ (c) $390°$ (d) $300°$
Write down the angle in degrees whose measurements in radians are:
(e) $3\pi/10$ (f) $11\pi/12$ (g) $13\pi/18$ (h) $27\pi/26$
What are:
(i) $\sin(\pi/2)$ (j) $\cos(\pi/3)$ (k) $\tan(3\pi/4)$ (l) $\cos 2\pi$
Simplify:
(m) $\sin(\pi/2 - \theta)$ (n) $\cos(\pi + \theta)$ (o) $\tan(2\pi - \theta)$
Notice that for the equations in problems (p), (q) and (r) there will be lots of solutions, and usually you will have to choose one of these by considering the other information given in the problem.
(p) If $5 \sin x = 4$ find a value of x between $\pi/2$ and π.
(q) If $\cos 3x = 1$ find a value of x between $-\pi$ and π.
(r) If $\tan 4x = 3$ find the values of x between 0 and 2π.
(s) Find the angle between 0 and 2π whose sine is 0.5. This is usually written:

$$\theta = \arcsin 0.5 \qquad \text{or} \qquad \theta = \sin^{-1} 0.5$$

(t) If $\sin\theta = 3/5$, and θ is acute, calculate $\cos\theta$ and $\tan\theta$.
(u) The population, N_L, of lynxes of a certain area of Canada varies sinusoidally with time and can be expressed by the approximate equation:

$$N_L = 40,000 + 35,000 \sin\left(\frac{2\pi t}{T}\right)$$

where T=11 years. Measurements were begun in January 1820 so take t to be zero at this date. What is the largest value the population ever reaches and what is the smallest? In what year (after 1820) does the population first have its largest value? In what year does it have its smallest value?
(v) Many problems call for functions which depend upon more than one variable. For instance a light wave or a sound wave propagated through time and space can be represented in a simplified form by:

$$y = A \sin\left(2\pi\left(\frac{x}{\lambda} + \omega t\right)\right)$$

where A is the amplitude, λ is the wavelength and ω is the frequency of the wave. x and t are position and time respectively. An understanding of this function is essential for

many problems (e.g. sound, light microscopy, phase microscopy, X-ray diffraction).
(I) Draw a graph of y as a function of x assuming $t = 0$.
(II) Draw a graph of y as a function of t assuming $x = 0$.
(III) At what values of x and t does the function repeat itself?
(IV) Draw the function $z = A \sin \frac{2\pi}{\lambda} (x + \alpha)$ where α is a constant.
(V) At what values of α are the curves $y = A \sin(2\pi/\lambda)$ and $z = A \sin(2\pi(x + \alpha/\lambda))$ in phase and out of phase?

Note: Make sure you know how to switch your calculator between degrees and radians.

Answers:
Exercise 1: (a) $3\pi/4$ (b) $3\pi/8$ (c) $13\pi/6$ (d) $5\pi/3$ (e) $54°$ (f) $165°$ (g) $130°$ (h) $186.9°$ (i) $+1$ (j) $+1/2$ (k) -1 (l) $+1$ (m) $\cos\theta$ (n) $-\cos\theta$ (o) $-\tan\theta$ (p) $x = 126.9°$ (q) $x = 0, \pm 2\pi/3$ (r) $x = 17.8 + 45n$ in $°$ where $0 > n > 8$ (s) $30°$ (t) $\cos\theta = 4/5$, $\tan\theta = 3/4$ (u) Largest value is when $\sin(2\pi t/T) = 1$, smallest value is when $\sin(2\pi t/T) = -1$, largest value is 75,000 in 1822 and smallest value is 5,000 in 1828.
(v) (III) Function repeats itself at $x = (1 - wt)\lambda$ and $t = (1 - \frac{x}{\lambda})/\omega$
(V) Values of α where the curves are in phase are $\alpha = 1 - \lambda x + n\lambda$, and out of phase are $\alpha = 1 - \lambda x + n\lambda + \lambda/2$.